B2.1 Introduction to Representation Theory
Problem Sheet 0, MT 2017

1. Recall the definition of modules and submodules over a ring R. Consider the vector space $V = \mathbb{C}^n$ of column vectors as a module over the ring $M_n(\mathbb{C})$ of n by n matrices. Show that the only submodules of V are $\{0\}$ and V. (A module with this property is called a *simple* module; such modules are the central object of study of this course.)

2. Recall the definition of minimal polynomial of a linear transformation. We denote by $GL(n, K)$ the group of invertible n by n matrices over a field K (with product rule being the usual multiplication of matrices).

   (a) Suppose $g \in GL(n, \mathbb{C})$ has finite order. Show that g is diagonalizable.

   (b) Suppose g is an element of finite order in $GL(n, K)$ where K has characteristic p. Must g be diagonalizable in that case? Show that if $n < p$ then g cannot have order p^2.

3. (a) Let G be a finite group. Show that there exists $n \in \mathbb{N}$ such that G is isomorphic to a subgroup of $GL(n, \mathbb{R})$.

   (b) [harder] Can you find a finite group H which cannot be isomorphic to a subgroup of $GL(2, \mathbb{C})$?